Analytical Mechanics: Worksheet 1

Degrees of freedom and holonomic constraints

Christophe De Beule (christophe.debeule@gmail.com)

---

1. The degrees of freedom of a mechanical system are the independent variables that completely determine its configuration. The dynamics of the system is known once their time dependence is known. Determine the degrees of freedom of:
   (a) A point mass in space;
   (b) A point mass moving in a plane;
   (c) A dumbbell (two point masses connected by a line of fixed length);
   (d) A rigid body consisting out of more than two point masses;
   (e) A spinning top with a fixed pivot.

2. Consider a chain made from M point masses connected by $M-1$ line rods where the joints can move freely. How many degrees of freedom does this system have? What if we close the chain? How do your conclusions change if the chain is confined to a plane?

3. Are the following holonomic constraints scleronomic (time independent; from Greek *skleros* "rigid" + *nomos* "law") or rheonomic (time dependent; from Greek *rheo* "to flow")?
   (a) A point mass sliding in a bowl;
   (b) A pendulum with a pivot that is moving up and down;
   (c) A spinning top in free fall;
   (d) A spinning top in an elevator.

4. Can the kinetic and potential energy depend explicitly on time for scleronomic constraints? What about rheonomic constraints? An interesting example is given by Problem 7.

5. Proof that the kinetic energy of a rheonomic system consisting of a point mass with one degree of freedom (q) can be written as

$$T(q,\dot{q},t) = \frac{1}{2}m\left[\dot{q}^2\left(\frac{\partial \vec{r}}{\partial q}\right)^2 + 2\dot{q}\frac{\partial \vec{r}}{\partial q}\cdot\frac{\partial \vec{r}}{\partial t} + \left(\frac{\partial \vec{r}}{\partial t}\right)^2\right].$$

6. Holonomic constraints restrict positions to a surface defined by algebraic equations involving only generalized coordinates q_k and t. Therefore, each particle's position is a smooth function $\vec{r} = \vec{r}(q_k, t)$. Use the chain rule to show that for a holonomic system:

$$\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_k} = \frac{\partial \vec{r}}{\partial q_k}.$$

7. Consider a bead that moves frictionless along a rod inclined at a fixed angle to the vertical axis. The rod rotates at constant angular velocity ω about the vertical axis in a uniform gravitational field, as illustrated in Figure 1.
   (a) How many degrees of freedom does this system have?
   (b) Are the constraints scleronomic or rheonomic? Write the constraints as $f(q,\phi,t) = 0$, where ϕ is the azimuthal angle.
   (c) Use spherical coordinates to express the position vector $\vec{r}$ of the bead and compute the kinetic energy $T = \frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}}$.

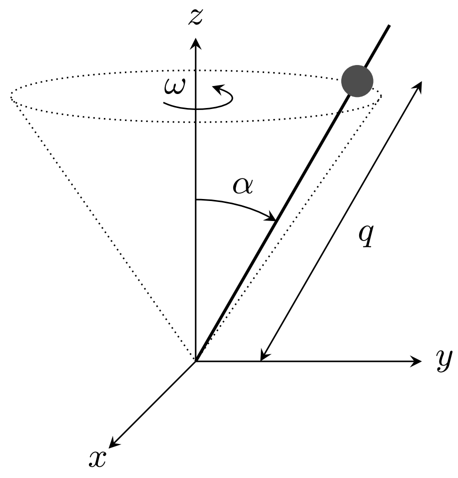

Figure 1: Bead on a rotating line.

(d) Calculate the virtual work $\delta W = \vec{F} \cdot \delta\vec{r}$ where

$$\delta\vec{r} \equiv \vec{r}\,(q+\delta q, t) - \vec{r}\,(q,t) = \frac{\partial \vec{r}}{\partial q}\delta q.$$

(e) Use the principal of d'Alembert to calculate the virtual work from $\delta W = \dot{\vec{p}} \cdot \delta\vec{r}$. To this end, write down $\vec{p} \cdot \delta\vec{r}$ and $\vec{p} \cdot \dot{\delta\vec{r}}$ in terms of partial derivatives of the kinetic energy. Hint: Use the result of Problem 6.

(f) Obtain the equations of motion by equating your two different results for δW.

(g) Give the general solution of the equations of motion by computing the homogeneous and particular solution. Write the integration constants in terms of $q(0)$ and $\dot{q}(0)$.

(h) Discuss the solution $q(t)$ for $\dot{q}(0) = 0$. In that case, what happens for $q(0) = g\cos\alpha/(\omega\sin\alpha)^2$. Is this a stable solution?