Analytical Mechanics: Worksheet 6

Hamiltonian mechanics

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1 Bead moving on a parabola

Consider a bead of mass m in a vertical plane that slides without friction along a parabolic wire. Choose the horizontal x as the generalized coordinate and take a parabola $y = x^2/2$.

(a) Determine the Lagrangian $L = L(x, \dot{x})$ and the Hamiltonian $H = H(x, p)$ with p the canonical momentum.

(b) Which quantity is conserved and why? How can the resulting equation be represented graphically in the phase space (x, p).

(c) Write down the Hamilton equations of motion. What do these express in relation to the graphical representation of the previous question.

2 Canonical transformation for a free-falling particle

The Hamiltonian is given by

$$H(q, p) = \frac{p^2}{2m} + mgq.$$

(a) Determine the trajectories in phase space (q, p) and make a sketch. Discuss how a particle moves along the trajectories using the Hamilton equations.

(b) Find a time-independent generating function $F\left(p, P\right)$ such that the transformed Hamiltonian $K\left(Q, P\right) = P$. Make use of

$$q = -\frac{\partial F}{\partial p}, \quad Q = \frac{\partial F}{\partial P}, \qquad K\left(Q, P\right) = H\left(q\left(Q, P\right), p\left(Q, P\right)\right).$$

First find an expression for $q(p, P)$.

(c) Explicitly determine the canonical transformation: find the functions $Q = Q(q, p)$ and $P = P(q, p)$.

(d) Show that Q is given by time up to constant.

(e) Use this result to find $p(t)$ and $q(t)$.

3 Hamilton-Jacobi equation for the harmonic oscillator

The Hamilton-Jacobi-vergelijking of the harmonic oscillator is given by

$$-\frac{\partial S}{\partial t} = H\left(q, \frac{\partial S}{\partial q}\right) = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2.$$

(a) Solve this equations using a separation of variables. Assume that the solution can be written as $S = W(q) - \alpha t$ where $P \equiv \alpha$ is a constant of motion. Write down the solution of $W(q)$ in terms of an integral.

(b) Note that the transformed momentum α does not appear in the transformed Hamiltonian $K = 0$. Hence,

$$Q \equiv \beta = \frac{\partial S}{\partial \alpha} = \text{constant.}$$

Make use of this expression to find the solution $q(t)$. Hint: $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x)$.

(c) Show that the total energy is given by α.