Analytical Mechanics: Worksheet 8

Relativistic kinematics

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1 Relativistic aberration

A light source in frame S emits a conical beam of light with opening angle θ such that at the edge $u^1 = c\cos\frac{\theta}{2}$ and $u^2 = c\sin\frac{\theta}{2}$. Find the opening angle θ' in a frame S' that is moving along the X-axis with speed V relative to S. Show that $\theta' \to \theta$ for $V/c \to 0$.

2 Spontaneous decay of a particle

A particle at rest with mass M and four momentum k decays spontaneously into two particles with mass m_1 and m_2 with four momenta p_1 and p_2. Write down the four momentum of each particle in the rest frame of the original particle. Then use conservation of four momentum to find the energy of both particles E_1 and E_2 in terms of M, m_1, and m_2.

3 Compton scattering

Compton scattering is the inelastic scattering of a photon with an electron. Denote the four momentum of the incoming and outgoing photon as k and k', and the four momentum of the electron before and after the collision as p and p'.

(a) Calculate the energy of the scattered photon in the rest frame of the incoming electron, in function of the photon scattering angle θ.

(b) Show that the maximal value for the energy of a back-scattered photon ($\theta = \pi$) in this frame is given by $\frac{1}{2}mc^2$ with m the electron mass.

4 Relativistic collision

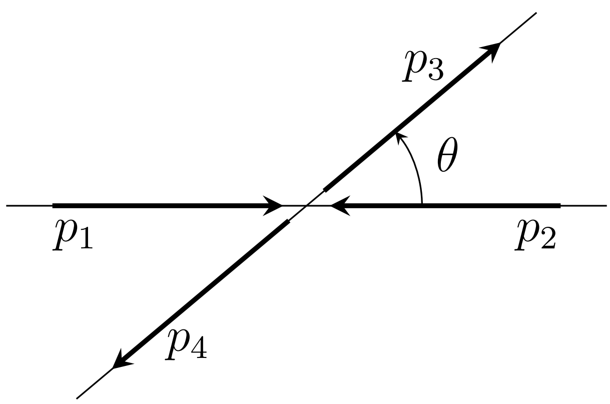

Figure 1: Scattering in the center-of-mass frame.

A particle with four momentum p_1 scatters at a second particle with four momentum p_2 that is at rest in the lab frame. Hence, in the lab frame

$$p_1^\mu = \left(\frac{E_1}{c}, p, 0\right), \qquad p_2^\mu = (m_2 c, 0, 0)\,.$$

In the center-of-mass frame, by definition, $\vec{P} = 0$ with $P = p_1 + p_2$ the total four momentum.

(a) Use the Lorentz invariance of P^2 to find the total energy in the center-of-mass frame.

(b) Calculate the speed β and Lorentz factor γ of to the Lorentz boost to the center-of-mass frame, and determine p_1 in this frame.

(c) Suppose that scattering is elastic such that the kinetic energy of each particle is conserved in the center-of-mass frame. Determine p_3 of one of the outgoing particles.

(d) Use the inverse Lorentz boost to find p_3 in the lab frame. Determine the scattering angle ψ in the lab frame.

(e) Write down the energy of the scattered particle p_3 in the lab frame as a function of β, γ, θ, and the initial energy. Discuss your results for $\theta = 0$.

(f) Use your results to compute the ratio of the kinetic energy $T_i = E_i - m_i c^2$ in the lab frame before and after the collision. Define $r = m_2/m_1$ and $\mathcal{E} = T_1/(m_1 c^2)$.

(g) Which scattering angle θ minimizes this ratio? Evaluate this case in the non-relativistic and ultra-relativistic limit.