Bosonization: Worksheet 3

Bosonic representation of free fermions

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1 Commutator of density operators

In the last exercise sheet, we saw that the density operator can be written as

$$\hat{\rho}_s(x) = \frac{1}{L}\sum_q e^{iqx}\,\hat{\rho}_{qs}, \qquad \hat{\rho}_{qs} = \sum_k \hat{c}^\dagger_{k-qs}\hat{c}_{ks} = \hat{\rho}^\dagger_{-qs},$$

where $\hat{\rho}_{qs}$ are the Fourier components of the density operator and we consider different flavors, denoted by s, of spinless fermions in *one spatial dimension*. Now consider the Hamiltonian

$$\hat{H}_0 = \hbar v_F \sum_{s=R,L}\sum_k (\epsilon_s k - k_F)\, {:}\hat{c}^\dagger_{ks}\hat{c}_{ks}{:},$$

where $v_F = \hbar k_F/m$, $k = \frac{2\pi}{L}n$ with $n \in \mathbb{Z}$ (periodic boundary conditions), the index s corresponds to left- or right-moving fermions with $\epsilon_s = \pm$ respectively, and :: indicates normal ordering with respect to the vacuum $|0\rangle_0$, which is defined as the state where all single-particle states with negative energy are occupied, i.e. all single-particle states with $\epsilon_s k \leq k_F$ are occupied (Dirac sea).

(1) Draw a picture of the spectrum for left and right movers. Sketch the ground state of $\hat{H}_0$ for $N_R = 4$ and $N_L = -2$ with N_s the number of s movers relative to the vacuum.

(2) Write the normal-ordered operator ${:}\hat{\rho}_{qs}{:} \equiv \hat{\rho}_{qs} - {}_0\langle 0|\,\hat{\rho}_{qs}\,|0\rangle_0$. Consider $q = 0$ and $q \neq 0$.

(3) Show that $[\hat{\rho}_{qs}, \hat{N}_{s'}] = 0$, where $\hat{N}_s = \sum_k {:}\hat{c}^\dagger_{ks}\hat{c}_{ks}{:}$ is the number operator (relative to the Dirac sea) for s movers. Interpret this result physically.

(4) Calculate the commutator $[\hat{\rho}_{qs}, \hat{\rho}^\dagger_{q's'}] = [\hat{\rho}_{qs}, \hat{\rho}_{-q's'}]$ for nonzero q and q'. Rewrite the commutator in terms of normal-ordered operators to avoid mistakes when subtracting infinities.

(5) Show that
$$\hat{\rho}_{q<0L}\,|0\rangle_0 = 0, \qquad \hat{\rho}_{q>0R}\,|0\rangle_0 = 0.$$

(6) Use the operators $\hat{\rho}_{qs}$ to define bosonic creation and destruction operators $\hat{b}^\dagger_q$ and $\hat{b}_q$ for $q \neq 0$ so that $\hat{b}_q\,|0\rangle_0 = 0$ and $[\hat{b}_q, \hat{b}^\dagger_{q'}] = \delta_{qq'}$. Consider $q > 0$ and $q < 0$ separately.

2 Bosonic representation of free Hamiltonian

Calculate the commutator $[\hat{H}_0, \hat{\rho}_{qs}]$, where $\hat{H}_0 = \sum_{s,k} E_{ks}\, {:}\hat{c}^\dagger_{ks}\hat{c}_{ks}{:}$ is a general non-interacting Hamiltonian. Now take $E_{ks} = \epsilon_s \hbar v_F k$ and show that in this case $\hat{H}_0$ can be written as

$$\hat{H}_0 = \frac{2\pi\hbar v_F}{L}\sum_{s=R,L}\sum_{q>0} {:}\hat{\rho}_{-qs}\hat{\rho}_{qs}{:} + \hat{E}_0,$$

where $[\hat{E}_0, \hat{\rho}_{qs}] = 0$. Determine $\hat{E}_0$ by computing the expectation value of the N-particle ground state $|N_L, N_R\rangle_0$ of $\hat{H}_0$ in both representations. This state contains $N = N_L + N_R$ more (or less) particles than $|0\rangle_0$. Finally, write $\hat{H}_0$ in terms of $\hat{b}^\dagger_q$ and $\hat{b}_q$.