Bosonization: Worksheet 5

Boson field operators

Christophe De Beule (christophe.debeule@gmail.com)

Starting from this exercise sheet, we use the following conventions. The density operator ρ_{qs} and corresponding commutator are defined as

$$\rho_{qs} = \sum_k c^\dagger_{k+qs} c_{ks}, \qquad [\rho_{qs}, \rho^\dagger_{q's'}] = -\delta_{ss'}\delta_{qq'}\epsilon_s \frac{qL}{2\pi},$$

Note that using $c^\dagger_{k-qs}$ instead of $c^\dagger_{k+qs}$ gives an extra minus sign in the commutator.

1 Boson field operators

Similar to the fermion field operators, we can define the boson field

$$\varphi(x) = \frac{\sqrt{\pi}}{L}\left(N_R + N_L\right) x + \frac{i\sqrt{\pi}}{L} \sum_{q\neq 0} \frac{1}{q}\, e^{-iqx - a|q|/2} \left(\rho_{qR} + \rho_{qL}\right),$$

where $a > 0$ is an infinitesimal regularization parameter. The boson field $\varphi(x)$ is chosen such that

$$\frac{1}{\sqrt{\pi}} \partial_x \varphi(x) = \rho_R(x) + \rho_L(x),$$

for $a \to 0$ and where $\rho_s(x) = :\psi^\dagger_s(x)\psi_s(x):$ with $\psi_s(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} c_{ks}$ and $N_s = \sum_k :c^\dagger_{ks} c_{ks}:$.

(1) Show that the equation for $\partial_x \varphi(x)$ is correct and also show that $\varphi(x)$ is Hermitian.

(2) Define another boson field, denoted as $\vartheta(x)$[1], in a similar manner as we defined $\varphi(x)$ and such that for $a \to 0$

$$\frac{1}{\sqrt{\pi}} \partial_x \vartheta(x) = \rho_R(x) - \rho_L(x).$$

(3) Show that

$$[\varphi(x), \vartheta(x')] = \frac{1}{2\pi}\left(\ln\left[1 - e^{i\frac{2\pi}{L}(x'-x+ia)}\right] - \ln\left[1 - e^{-i\frac{2\pi}{L}(x'-x-ia)}\right]\right).$$

Hint: $\sum_{n=1}^\infty z^n/n = -\ln(1-z)$ for $|z| < 1$.

(4) Show that

$$[\varphi(x), \vartheta(x')] \xrightarrow{L\to\infty,\ a\to 0} \begin{cases} \frac{i}{2}\,\mathrm{sgn}(x - x') & x \neq x', \\ 0 & x = x'. \end{cases}$$

Hint: Keep terms in $1/L$ and note that the complex logarithm is defined as $\ln z \equiv \ln|z| + i \arg z$ for $z \neq 0$ and $\arg z \in (-\pi, \pi]$, i.e. we put the branch cut along the negative real axis.

(5) Use this result to show that

$$[\varphi(x), \partial_{x'}\vartheta(x')] \xrightarrow{L\to\infty,\ a\to 0} -i\delta(x - x').$$

Note that if we define $\Pi(x) = -\partial_x \vartheta(x)$, we obtain the canonical form $[\varphi(x), \Pi(x')] = i\delta(x - x')$. Hint: $\delta(x) = \partial_x \Theta(x)$.

---

[1] The boson field $\vartheta(x)$ is not to be confused with the step function $\Theta(x)$.

(6) Now start from (3) and take the derivative with respect to x'. Show that

$$[\varphi(x), \partial_{x'}\vartheta(x')] = -i\left[\frac{a/\pi}{(x'-x)^2+a^2} - \frac{1}{L}\right] + \mathcal{O}(a/L^2),$$

and show that you obtain the same result as (5) in the limits $L \to \infty$ and $a \to 0$.

(7) Integrate your result from (6) with respect to x' to find

$$[\varphi(x), \vartheta(x')] = C + \frac{i}{\pi}\arctan\left(\frac{x-x'}{a}\right) + \frac{ix'}{L},$$

where we only kept the lowest-order terms and C is an integration constant. Determine C by requiring that $[\varphi(x), \vartheta(x)] = 0$. Finally take the limits $L \to \infty$ and $a \to 0$ to obtain the result in (4). Note that for finite a and L, the sign function is smeared over a range of order a and that there is a term of order $1/L$.

2 Free Hamiltonian in terms of boson fields

Remember that the free Hamiltonian could be written as

$$H_0 = \frac{\pi v_F}{L}\sum_s \sum_{q\neq 0} :\rho_{-qs}\rho_{qs}: + \frac{\pi v_F}{L}\sum_s N_s^2,$$

where we put $\hbar = 1$. Note that the second term is different than in worksheet 3. This is because we are now using anti-periodic boundary conditions instead of periodic boundary conditions. The type of boundary condition makes no difference in the limit $L \to \infty$.

(1) Show that

$$\frac{1}{2}\int_{-L/2}^{L/2} dx\,(\partial_x\varphi(x))^2 = \frac{\pi}{2L}\sum_{q\neq 0}(\rho_{-qR}+\rho_{-qL})(\rho_{qR}+\rho_{qL}) + \frac{\pi}{2L}(N_R+N_L)^2,$$

for $a \to 0$. Hint: Note that $q = \frac{2\pi}{L}n$ ($n \in \mathbb{Z}$), independent of the boundary conditions.

(2) Perform the same calculation for the boson field $\vartheta(x)$ and show that

$$H_0 = \frac{v_F}{2}\int_{-L/2}^{L/2} dx :\Big[(\partial_x\varphi(x))^2 + (\partial_x\vartheta(x))^2\Big]:,$$

which only holds for anti-periodic boundary conditions if we do not take the limit $L \to \infty$.

(3) Finally, show that H_0 can also be written as

$$H_0 = \pi v_F \int_{-L/2}^{L/2} dx :\big[\rho_R^2(x) + \rho_L^2(x)\big]: .$$

We need normal ordering in the expressions above, because even though $\partial_x\varphi(x)$ and $\partial_x\vartheta(x)$ are normal ordered, their squares are not in general. Indeed, $\langle :A: \rangle_0 = 0$ but $\langle :A::A: \rangle_0 = \langle A^2\rangle_0 - (\langle A\rangle_0)^2$.