Bosonization: Worksheet 6

Boson representation of fermion fields

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1 Density operator

Use the general definition of the fermion field operator $\psi(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} c_k$, i.e. before linearizing the spectrum, to show that

$$\begin{aligned}\rho(x) &\equiv \psi^\dagger(x)\psi(x) \\ &\simeq \rho_R(x) + \rho_L(x) + \psi_R^\dagger(x)\psi_L(x) + \psi_L^\dagger(x)\psi_R(x),\end{aligned}$$

where $\simeq$ indicates that we only consider states close to the Fermi energy. Write down each term explicitly in terms of sums over restricted momenta.

2 Boson representation of fermion fields

Remember that

$$\rho_{qs} = \sum_k c^\dagger_{k+qs} c_{ks}, \qquad [\rho_{qs}, \rho^\dagger_{q's'}] = -\delta_{ss'}\delta_{qq'}\epsilon_s \frac{qL}{2\pi},$$

where ϵ_s is given by ± 1 for $s = R, L$, respectively.

(1) Show that

$$[\rho_{qs}, \psi_s(x)] = -e^{iqx}\psi_s(x),$$

which suggests that $\psi_s(x) = f(\{\rho^\dagger_{qs}\})$ which is some function of the $\rho^\dagger_{qs}$.

(2) Use the fundamental commutator of bosonization to show that

$$[\rho_{qs}, (\rho^\dagger_{q's})^n] = -n\delta_{qq'}\frac{\epsilon_s qL}{2\pi},$$

which can be proven by induction. Using the ansatz for $\psi_s(x)$, where f is defined through its Taylor series, show that this implies

$$[\rho_{qs}, \psi_s(x)] = -\frac{\epsilon_s qL}{2\pi}\frac{\partial f}{\partial \rho^\dagger_{qs}}.$$

(3) Combine the results of (1) and (2) to show that

$$\psi_s(x) = (2\pi a)^{-1/2}\, U_s\, \lambda_s(x)\, e^{\epsilon_s \frac{2\pi}{L}\sum_{q\neq 0}\frac{1}{q}e^{iqx - a|q|/2}\rho^\dagger_{qs}},$$

where U_s commutes with all ρ_{qs} for $q \neq 0$ and $\lambda_s(x)$ is to be determined. The operator U_s is called the Klein factor and reduces the number of s movers by one. We have also added the regularization parameter a, as the terms in the exponent are not normal ordered.

(4) Let us find the normal-ordered form. Take $s = R$ and use $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ to show that

$$e^{\frac{2\pi}{L}\sum_{q\neq 0}\frac{1}{q}e^{iqx-a|q|/2}\rho^\dagger_{qR}} = \left(\frac{2\pi a}{L}\right)^{1/2} e^{\frac{2\pi}{L}\sum_{q<0}\frac{1}{q}e^{iqx}\rho^\dagger_{qR}} e^{\frac{2\pi}{L}\sum_{q>0}\frac{1}{q}e^{iqx}\rho^\dagger_{qR}},$$

and therefore

$$\psi_R(x) = \frac{U_R}{\sqrt{L}}\,\lambda_R(x)\, e^{\frac{2\pi}{L}\sum_{q<0}\frac{1}{q}e^{iqx}\rho^\dagger_{qR}} e^{\frac{2\pi}{L}\sum_{q>0}\frac{1}{q}e^{iqx}\rho^\dagger_{qR}},$$

where we put $a = 0$ on the right-hand sides because the exponents are normal ordered, since $\rho^\dagger_{q>0R}\,|0\rangle = 0$. In general, we have

$$\psi_s(x) = \frac{U_s}{\sqrt{L}}\,\lambda_s(x)\,e^{-i\phi_s^\dagger(x)}\,e^{-i\phi_s(x)}, \qquad \phi_s(x) = i\frac{2\pi}{L}\sum_{\epsilon_s q>0}\frac{1}{q}\,e^{iqx-a|q|/2}\rho^\dagger_{qs}.$$

(5) Calculate ${}_0\langle N_s|\,U_s^\dagger\psi_s(x)\,|N_s\rangle_0$ in two different ways, to show that

$$\lambda_s(x) = e^{i\frac{2\pi}{L}\left(N_s-\frac{1}{2}\right)x},$$

where we define U_s such that $U_s^\dagger U_s = 1$ and when acting on the ground state with N_s particles, U_s removes the highest occupied s-level.

(6) Show that $[N_s, \psi_s(x)] \propto [N_s, U_s]$ and use (1) to show that

$$[N_s, U_s] = -U_s,$$

which indeed shows that U_s reduces the particle number: $N_s U_s = U_s\,(N_s - 1)$.

3 Checking anti-commutators

(1) Show that $[\phi_s(x), \phi_{s'}(x')] = [\phi_s^\dagger(x), \phi_{s'}^\dagger(x')] = 0$ and that

$$[\phi_s(x), \phi_{s'}^\dagger(x')] = -\delta_{ss'}\ln\left[1 - ye^{-2\pi a/L}\right],$$

with $y = e^{i\frac{2\pi}{L}\epsilon_s(x-x')}$.

(2) Use this result together with the formula $e^A e^B = e^B e^A e^{[A,B]}$ (see worksheet 1) to show that

$$e^{-i\phi_s(x)}e^{-i\phi_s^\dagger(x')} = e^{-i\phi_s^\dagger(x')}e^{-i\phi_s(x)}\left(1 - ye^{-2\pi a/L}\right).$$

(3) Show that $e^{\alpha N_s}U_s = U_s e^{\alpha(N_s-1)}$ with $\alpha \in \mathbb{R}$ and use the normal-ordered form

$$\psi_s(x) = \frac{U_s}{\sqrt{L}}\,e^{i\frac{2\pi}{L}\left(N_s-\frac{1}{2}\right)x}\,e^{-i\phi_s^\dagger(x)}\,e^{-i\phi_s(x)},$$

to show that in the limit $a \to 0$

$$\{\psi_s(x), \psi_s(x')\} = 0,$$

and

$$\{\psi_s(x), \psi_s^\dagger(x')\} = O(x, x')\,\frac{a/\pi}{(x-x')^2 + a^2} \overset{a\to 0}{=} \delta(x - x'),$$

in the limit $L \to \infty$, where $O(x, x) = 1$.