Quantum Mechanics

Sample exam 1

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Question 1 – Perturbation theory

Consider a two-dimensional harmonic oscillator with the perturbation $V = x^2 y^2$.

(a) Find the energy correction to the second-excited state up to first order, together with the corresponding eigenstates. Draw an energy diagram.

(b) Now consider the perturbation $V = x^2 f(y)$, where $f(y)$ is an *analytic* (operator-valued) function. Show that there is no first-order correction for the second-excited state if $f(y)$ is an odd function, i.e. for $f(-y) = -f(y)$.

Recall that the position operator can be written as

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right),$$

where a and $a^\dagger$ are the annihilation and creation operators, for which

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

Question 2 – Born approximation

The spherical potential well is given by

$$V(r) = \begin{cases} V_0 & r \le a \\ 0 & r > a. \end{cases}$$

(a) Calculate the scattering amplitude in the (first) Born approximation.

(b) Expand your result for small qa up to second order.

(c) Write down the differential cross section and use your approximate result to find the total cross section in the low-energy limit as a function of ka with $k = \sqrt{2mE}/\hbar$.

Question 3 – Partial waves

Scattering resonances of the delta-shell potential,

$$V(r) = \lambda\delta(r-a).$$

(a) Consider s-waves ($l = 0$) and find $u(r) = rR_0(r)$ inside and outside the shell. Assume that $u(r)$ is continuous and find the other boundary condition. Use these to determine the s-wave phase shift δ_0. Show that you recover the result of the hard sphere if $\mu \equiv 2m\lambda/\hbar^2 k \gg 1 - \cot ka$, where k is the length of the incident wave vector.

(b) Consider δ_0 as a function of a while keeping k fixed for different values of μ and find the condition for resonances ($\delta_0 = \pi/2$).

(c) Determine the s-wave bound states of the infinite spherical potential *well* of radius a and discuss their relationship with the delta-shell resonances in case $\mu \gg 1$.

Formulas

Series expansions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7),$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \mathcal{O}(x^6).$$

Trigonometric identities:

$$\cos^2 x - \sin^2 x = \cos 2x,$$

$$2 \cos x \sin x = \sin 2x.$$

For a spherically-symmetric potential, the radial Schrödinger equation for angular momentum quantum number l, is given by

$$\left[\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2} \left(E - V(r) \right) \right] R_l(r) = 0.$$