Quantum Mechanics

Sample exam 3

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Question 1 – Perturbation theory

Consider the harmonic oscillator

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2,$$

and the relativistic kinetic energy correction

$$H' = -\frac{p^4}{8m^3c^2}.$$

(a) Find the first-order correction due to H' to the ground state of the one-dimensional harmonic oscillator.

(b) Use your result to find the first-order correction to the ground state of the harmonic oscillator in two and three spatial dimensions.

Recall that the momentum operator can be written as

$$p_x = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^\dagger - a\right),$$

where a and $a^\dagger$ are the annihilation and creation operators, for which

$$\begin{aligned} a\left|n\right\rangle &= \sqrt{n}\left|n-1\right\rangle \\ a^\dagger\left|n\right\rangle &= \sqrt{n+1}\left|n+1\right\rangle. \end{aligned}$$

Question 2 – Variational method

Use a proper variational wave function that depends on a single parameter α, and find the best approximation to the ground-state energy of a particle moving in one spatial dimension in a quartic potential. The Hamiltonian is

$$H = \frac{p^2}{2m} + Ax^4.$$

Question 3 – Partial waves

Consider s-wave scattering from an attractive potential shell ($V_0 > 0$)

$$V(r) = \begin{cases} -V_0 & a \leq r \leq b \\ 0 & \text{elsewhere.} \end{cases}$$

Remember that the radial Schrödinger equation for $l = 0$ is given by

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_0}{dr}\right) + \frac{2m}{\hbar^2}\left(E - V(r)\right)R_0 = 0,$$

where $R_0(r)$ is the radial wave function.

(a) Find the general solution in the three different regions. Hint: use the substitution $u(r) = rR_0(r)$ to simplify the radial equation.

(b) Use the boundary conditions to find an expression for $\tan(kb + \delta_0)$.

(c) Show that you obtain the result for the spherical potential well $V(r) = -V_0\theta(b - r)$ in the limit $a \to 0$, given by

$$\lim_{a \to 0} \delta_0 = -kb + \arctan\left(\frac{k \tan qb}{q}\right).$$