Quantum Mechanics: Worksheet 2

Degenerate perturbation theory

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1 Two-dimensional square well

Consider a particle in a two-dimensional infinite potential well:

$$V(x,y) = \begin{cases} 0 & 0 < x, y < a \\ \infty & \text{elsewhere,} \end{cases}$$

perturbed by a finite square well in the lower-left corner:

$$W(x,y) = \begin{cases} W_0 & 0 < x, y < a/2 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Draw the system and write down the energy and the eigenstates of the unperturbed Hamiltonian. Determine the degeneracy of the first three energy levels.

(b) Calculate the first-order correction to the energy of the ground state and the first excited state and draw an energy-level diagram.

(c) How do the two degenerate wave functions (doublet) of the first excited state transform into each other under reflection about the diagonal? Do the same for the new basis that diagonalizes the perturbation matrix.

The original doublet, the two degenerate states of the first excited level, break up into two singlets under the reduced symmetry ($C_{4v} \to C_2$) of the perturbed system. This is an example of degeneracy being lifted by symmetry-breaking perturbations.

2 Two-dimensional harmonic oscillator with xy-coupling

Consider the Hamiltonian $H = H_0 + \lambda V$, where

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m\omega^2 \left(x^2 + y^2\right), \quad V = m\omega^2 xy.$$

(a) Express H_0 in terms of ladder operators $(a^\dagger, a)$ and $(b^\dagger, b)$, which correspond to the x and y coordinates, respectively.

(b) Write down the energy and the eigenstates of the unperturbed system. Determine the degeneracy of the first three energy levels.

(c) Calculate the lowest-order correction to the energy of the ground state, the first excited state, and the second excited state. Draw an energy-level diagram.

Exact solution To solve this problem exactly we have to make a change of variables to decouple the oscillators.

(1) Rewrite the total potential of the oscillator system as a function of $x + y$ and $x - y$.

(2) Define new variables x', y' so that the potential can be written as

$$\frac{1}{2}m\omega_x'^2 x'^2 + \frac{1}{2}m\omega_y'^2 y'^2,$$

and determine the oscillator frequencies ω_x' and ω_y'.

(3) Find the corresponding canonical momenta p_x' and p_y' and write down the Hamiltonian with the new variables. Determine the exact energy eigenvalues.

(4) Compare your result with the perturbative results by expanding the exact spectrum up to second order in λ.

(5) Consider the case $\lambda = 1$ and $\lambda = -1$. What happens for $|\lambda| > 1$?