Quantum Mechanics: Worksheet 4

Scattering theory: Green's operator and the Born approximation

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1 Green's operator

The free Green's operator $\hat{G}_0(z)$ (resolvent) is defined by the operator equation

$$(z - \hat{H}_0)\hat{G}_0(z) = 1,$$

where z is a complex parameter.

(a) Fourier transform the retarded Green's operator $\hat{G}_0^R(E) = \hat{G}_0(E + i\epsilon)$ to the time domain, i.e., calculate

$$\hat{G}_0^R(t) = \frac{1}{2\pi\hbar} \int dE\, \hat{G}_0^R(E) \exp\left(-\frac{i}{\hbar}Et\right).$$

(b) Show that $\hat{G}_0^R(t)$ satisfies

$$\left(i\hbar\frac{d}{dt} - \hat{H}_0\right) \hat{G}_0^R(t) = \delta(t), \qquad \hat{G}_0^R(t)\Big|_{t<0} = 0.$$

(c) Consider $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$ and show that $\hat{G}_0^R(t)$ evolves $|\psi(t)\rangle$ forward in time, where $|\psi(t)\rangle$ satisfies

$$\left(i\hbar\frac{d}{dt} - \hat{H}_0\right) |\psi(t)\rangle = \hat{V}(t)\, |\psi(t)\rangle\,,$$

with the condition $|\psi(t)\rangle = |\psi_0(t)\rangle$ for $\hat{V} = 0$.

(d) Perform the same calculations for the advanced Green's operator $\hat{G}_0^A(E) = \hat{G}_0(E - i\epsilon)$ and show that $\hat{G}_0^A(t) = \hat{G}_0^R(-t)^\dagger$.

2 Scattering cross-section in the Born approximation

In an elastic scattering process, we consider an incident plane wave $|\boldsymbol{k}\rangle$ that is scattered by a static potential $V(\boldsymbol{r})$ into some direction $\hat{\boldsymbol{n}}(\theta, \phi)$ so that the outgoing wave vector is given by $\boldsymbol{k}' = k\hat{\boldsymbol{n}}$. This is illustrated in Figure 1. The probability for a particle to scatter into this direction is given by the differential cross-section:

$$\frac{d\sigma}{d\Omega}(\theta, \phi) = |f(\boldsymbol{k}', \boldsymbol{k})|^2,$$

with $f(\boldsymbol{k}', \boldsymbol{k})$ the scattering amplitude. In the Born approximation, the scattering amplitude is approximated by

$$\begin{aligned} f(\boldsymbol{k}', \boldsymbol{k}) \simeq f^{(1)}(\boldsymbol{k}', \boldsymbol{k}) &= -\frac{m}{2\pi\hbar^2}(2\pi)^3\langle\boldsymbol{k}'|V|\boldsymbol{k}\rangle \\ &= -\frac{m}{2\pi\hbar^2} \int d^3\boldsymbol{r}\, V(\boldsymbol{r}) e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}, \end{aligned}$$

with $\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$. Calculate the differential cross-section in the Born approximation for the following spherically-symmetric scattering potentials ($b > 0$):

(a) $V(\boldsymbol{r}) = 4\pi V_0 \delta(\boldsymbol{r}/b)$

(b) $V(r) = \frac{1}{2}V_0 e^{-r/b}$

(c) $V(r) = \frac{4}{\sqrt{\pi}}V_0 e^{-(r/b)^2}$

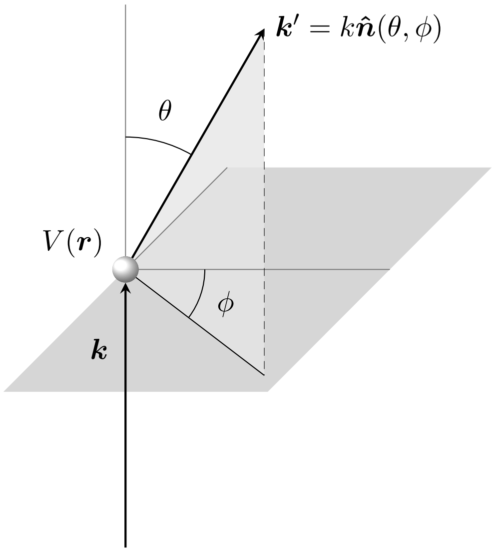

Figure 1: Illustration of an elastic scattering process.

3 Rutherford scattering at a hydrogen atom in the ground state

Consider an incident particle with mass m and charge Ze that interacts with a hydrogen *atom* in the ground state. The potential is given by (in Gaussian units)

$$V(r) = Ze^2 \left(\frac{1}{r} - \int d^3\boldsymbol{r}' \frac{\rho(r')}{|\boldsymbol{r} - \boldsymbol{r}'|} \right),$$

where $\rho(r)$ is the density of the atomic electron:

$$\rho(r) = |\psi(r)|^2 = \frac{1}{\pi a_0^3} e^{-2r/a_0},$$

with $a_0 = \frac{\hbar^2}{m_e e^2}$ the Bohr radius.

(a) Evaluate the scattering potential far away from the atom, i.e. for $r \gg a_0$. It is best to take the limit before you perform the integration to simplify the calculation.

(b) Write the scattering amplitude in the first Born approximation as

$$f^{(1)}(\boldsymbol{k}', \boldsymbol{k}) = -\frac{2m}{\hbar^2} \frac{Ze^2}{q^2} F(q),$$

where $q = |\boldsymbol{k}' - \boldsymbol{k}|$ and $F(q)$ is called the atomic form factor. Regularize the Fourier transform of the bare Coulomb potential by first substituting $\frac{1}{r}$ with the Yukawa potential $\frac{e^{-\mu r}}{r}$ and put $\mu = 0$ at the end of the calculation.

(c) Calculate the form factor and expand it near $q = 0$ up to second order. Try to explain physically why the the scattering amplitude at $q = 0$ is no longer divergent as is the case for the bare Coulomb potential.

(d) Write down the differential cross-section.

(e) Calculate the total cross-section and discuss your result for $ka \ll 1$ (low-energy scattering) and $ka \gg 1$ (high-energy scattering).