Topological Systems: Worksheet 1

Square lattice in a magnetic field

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Consider the Hamiltonian of electrons hopping on a square lattice with a single orbital per site is given by,

$$\hat{H} = \sum_{m,n} t_x c^\dagger_{m+1,n} c_{m,n} + t_y c^\dagger_{m,n+1} c_{m,n} + h.c., \tag{1}$$

where m and n label the sites and t_x and t_y are hopping amplitudes in the x and y direction, as illustrated in Fig. 1 (a). The Hamiltonian (1) can be diagonalized by a Fourier transform, giving a single energy band with $E(k_x, k_y) = 2t_x \cos k_x a + 2t_y \cos k_y a$ where a is the lattice constant. To introduce a magnetic field in the lattice model, we perform the Peierls substitution

$$t c^\dagger_{\boldsymbol{r}'} c_{\boldsymbol{r}} \to t \exp\left(-\frac{2\pi i}{\phi_0} \int_{L(\boldsymbol{r},\boldsymbol{r}')} d\boldsymbol{l} \cdot \boldsymbol{A} \right) c^\dagger_{\boldsymbol{r}'} c_{\boldsymbol{r}},$$

where the path $L(\boldsymbol{r}, \boldsymbol{r}')$ goes along a straight line from the site at $\boldsymbol{r}$ to the site at $\boldsymbol{r}'$, with $\phi_0 = h/e$ the flux quantum and where $\boldsymbol{A}$ is the vector potential with $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

(1) We now consider a magnetic field perpendicular to the xy-plane of the square lattice. Use the Landau gauge $\boldsymbol{A} = -By\boldsymbol{e}_x$ and calculate the Peierls phase for the square lattice along the x and y direction. How are the hopping amplitudes t_x and t_y affected by the magnetic field?

(2) Write down the Hamiltonian for the gauge we have chosen. Is the Hamiltonian still periodic in x and y?

(3) What is the condition on the flux through a square plaquette $\phi = Ba^2$ such that the Hamiltonian becomes periodic again. Such a flux, for which the periodicity is restored, is called a commensurate flux. How does this affect the unit cell?

(4) Perform a Fourier transform on the Hamiltonian

$$c_{m,j,\alpha} = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}} e^{i(k_x m + k_y j q)a} c_{\boldsymbol{k},\alpha},$$

where j and n label the magnetic unit cells with $n = j + \alpha$. Here, α labels the sites in the magnetic unit cell and $q > 0$ is an integer. How is q connected to the magnetic flux ϕ and what is the range of unique values of k_x and k_y, i.e. what constitutes the magnetic Brillouin zone? Hint: first perform the Fourier transform along the x direction.

(5) Show that the Hamiltonian can be written as

$$\hat{H} = \sum_{\boldsymbol{k}} \sum_{\alpha\alpha'} h_{\alpha\alpha'}(\boldsymbol{k}) c^\dagger_{\boldsymbol{k}\alpha} c_{\boldsymbol{k}\alpha'},$$

where $h_{\alpha\alpha'}(\boldsymbol{k})$ is the magnetic Bloch Hamiltonian. Find the explicit form of the magnetic Bloch Hamiltonian for $q = 2$.

(6) How many energy bands are there for a given commensurate flux? What charge density can each magnetic band hold?

(7) The support of the magnetic band structure, i.e. allowed values of $(E, \phi/\phi_0)$, gives rise to a fractal pattern called the Hofstadter butterfly. (optional) Numerically calculate the Hofstadter butterfly of the square lattice for $\boldsymbol{k} = 0$ and several values of t_x/t_y.

(8) Consider the case $q = 2$ and calculate the magnetic bands. Show that the system is gapless and has Dirac cones near $k_x a = \pm\pi/2$ and $k_y a = \pi/2$.

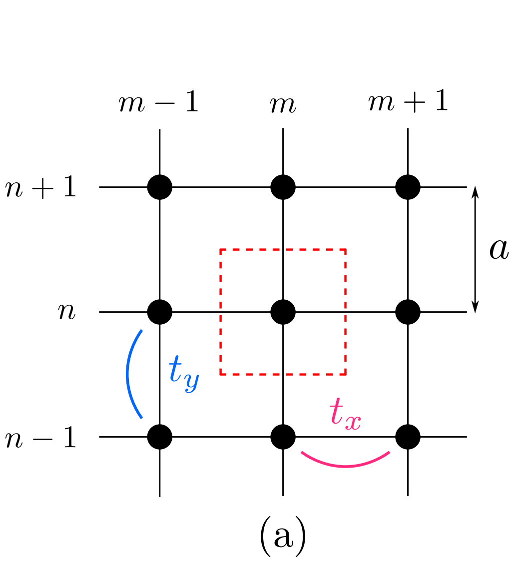

Figure 1: (a) Square lattice for zero magnetic field where the unit cell is denoted by the dashed red lines. (b) Square lattice for magnetic flux $\phi/\phi_0 = l + 1/2$ ($l \in \mathbb{Z}$) per square plaquette.